WITTEN'S CONJECTURE FOR MANY FOUR-MANIFOLDS OF SIMPLE TYPE

PAUL M. N. FEEHAN AND THOMAS G. LENESS

Abstract. We show how Witten's conjecture relating the Donaldson and Seiberg-Witten series for four-manifolds of Seiberg-Witten simple type with $b_1 = 0$, odd $b_2^+ \geq 3$ and $c_1^2(X) \geq \chi_h(X) - 3$ or which are abundant follows from the SO(3) monopole cobordism formula.

1. Introduction

1.1. **Main results.** Throughout this article we shall assume that X is a *standard* four-manifold by which we mean that X is smooth, connected, closed, and oriented with $b_1(X) = 0$ and odd $b^+(X) \geq 3$. For such manifolds, we define (by analogy with their values when X is a complex surface)

$$c_1^2(X) = 2\chi + 3\sigma, \quad \text{and} \quad \chi_h(X) = \tfrac{1}{4}(\chi + \sigma), \tag{1.1}$$

where χ and σ are the Euler characteristic and signature of X.

For standard four-manifolds, the Seiberg-Witten invariants [24], [29], [32] comprise a function with finite support, $SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$, where $\mathrm{Spin}^c(X)$ is the set of isomorphism classes of spin^c structures on X. The set of SW-basic classes, $B(X)$, is the image under a map $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ of the support of SW_X, [32]. A standard four-manifold X has Seiberg-Witten simple type if $c_1^2(\mathfrak{s}) = c_1^2(X)$ for all $c_1(\mathfrak{s}) \in B(X)$ and is *abundant* if $B(X)^\perp$ contains a hyperbolic summand.

Let $\mathbf{D}_X^w(h)$ denote the Donaldson series (see [21, Theorem 1.7] or §2.2 here). Using arguments from quantum field theory, Witten made the following conjecture.

Conjecture 1.1 (Witten's Conjecture)**.** Let X be a standard four-manifold with Seiberg-Witten simple type. Then X has Kronheimer-Mrowka simple type and the Kronheimer-Mrowka and Seiberg-Witten basic classes coincide. For any $w \in H^2(X;\mathbb{Z})$ and $h \in H_2(X;\mathbb{R})$, if Q_X is the intersection form of X,

$$\mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h\rangle}. \tag{1.2}$$

In [7], we showed that a formula, given here as Theorem 3.1, relating Donaldson and Seiberg-Witten invariants followed from certain properties, described in Remark 3.2, of the gluing map for SO(3) monopoles constructed in [5]. A complete proof of these properties is the subject of [5] and [6]. The formula in Theorem 3.1 involves polynomials with unknown

Date: This version: April 6, 2009. math.DG/yymmnnn.

PMNF was supported in part by NSF grants DMS 9704174 and DMS 0125170. TGL was supported in part by a Florida International University Summer Research Grant.

coefficients depending on topological data and thus lacks the simplicity and explicit form of Conjecture 1.1. In this paper, we use a family of manifolds constructed by Fintushel, J. Park, and Stern in [16] to determine enough of these coefficients to prove the following.

Theorem 1.2. *Theorem 3.1 implies that any standard four-manifold which is abundant or which satisfies $c_1^2(X) \geq \chi_h(X) + 3$ satisfies Conjecture 1.1.*

The quantum field theory argument giving equation (1.2) for standard four-manifolds has been extended by Moore and Witten [23] to allow $b^+(X) \geq 1$, $b_1(X) \geq 0$, and four-manifolds X of non-simple type. The SO(3) monopole program gives a relation between the Donaldson and Seiberg-Witten invariants for these manifolds and so should also lead to a proof of Moore and Witten's more general conjecture. However, the methods of this paper do not extend to the more general case because of the lack of examples of four-manifolds not of simple type.

A proof of Witten's conjecture, also assuming Theorem 3.1, for a more restricted class of manifolds has appeared previously in [22, Corollary 7]. Theorem 1.2 holds for all simply-connected, minimal algebraic surfaces of general type. An earlier version of this paper claimed Theorem 1.2 held for all standard four-manifolds; we explain the gap in that argument in Remark 4.9.

1.2. **Outline of the article.** In [7], we showed that a Donaldson invariant of a four-manifold X could be expressed as a polynomial $p(X)$ in the intersection form of X, the Seiberg-Witten basic classes of X and an additional $\Lambda \in H^2(X;\mathbb{Z})$ which does not appear in (1.2). If X has SW-simple type, then the coefficients of $p(X)$ depend only on the degree of the Donaldson invariant, Λ^2, $\chi_h(X)$, $c_1^2(X)$, and $c_1(\mathfrak{s}) \cdot \Lambda$ for $c_1(\mathfrak{s})$ an SW-basic class. We prove Theorem 1.2 by using examples of manifolds known to satisfy Conjecture 1.1 to determine some of these coefficients.

In §2, we review the definition of the Donaldson series, the Seiberg-Witten invariants and results on the surgical operations of blowing up, blowing down, and rational blow-downs which preserve (1.2). In §3, we summarize the results of the SO(3) monopole cobordism program from [7], giving the equality between the Donaldson invariant and the polynomial $p(X)$ mentioned above. The proof of Theorem 1.2 appears in §4.

1.3. **Acknowledgements.** The second author is indebted to Ron Stern and Ron Fintushel for much help with examples, to Nick Saveliev for comments on drafts, and to Miroslav Yotov for Lemma 4.1. In addition, the second author would like to thank the organizers of the Park City Mathematics Institute 2006 Summer School for providing an outstanding research environment and Tom Mrowka in particular for his encouragement of this project.

2. Preliminaries

We begin by reviewing the relevant properties of the Donaldson and Seiberg-Witten invariants.

2.1. **Seiberg-Witten invariants.** As stated in the introduction, the Seiberg-Witten invariants defined in [32] (see also [24, 28, 29]), define a map with finite support,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

where $\mathrm{Spin}^c(X)$ denotes the set of spin^c structures on X. For a spin^c structure $\mathfrak{s} = (W^{\pm}, \rho)$ where $W^{\pm} \to X$ are complex rank-two bundles and ρ is a Clifford multiplication map, define $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ by $c_1(\mathfrak{s}) = c_1(W^+)$. For all $\mathfrak{s} \in \mathrm{Spin}^c(X)$, $c_1(\mathfrak{s})$ is characteristic.

The invariant $SW_X(\mathfrak{s})$ is defined by the homology class of $M_{\mathfrak{s}}$, the moduli space of Seiberg-Witten monopoles. A *SW-basic class* is $c_1(\mathfrak{s})$ where $SW_X(\mathfrak{s}) \neq 0$. Define

$$B(X) = \{c_1(\mathfrak{s}) : SW_X(\mathfrak{s}) \neq 0\}. \tag{2.1}$$

If $H^2(X;\mathbb{Z})$ has two-torsion, then $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ is not injective. The formulas in this paper are equalities in real homology and cohomology so we define

$$SW'_X : H^2(X;\mathbb{Z}) \to \mathbb{Z}, \quad K \mapsto \sum_{\mathfrak{s}\in c_1^{-1}(K)} SW_X(\mathfrak{s}), \tag{2.2}$$

and set $SW_X(K) = 0$ if K is not characteristic. With this definition, (1.2) is equivalent to

$$\mathbf{D}^w_X(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w^2+K\cdot w)} SW'_X(K) e^{\langle K,h\rangle}. \tag{2.3}$$

A manifold X has *SW-simple type* if $SW_X(\mathfrak{s}) \neq 0$ implies that $c_1^2(\mathfrak{s}) = c_1^2(X)$.

As discussed in [24, §6.8], there is an involution on $\mathrm{Spin}^c(X)$, $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$, with $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$, defined essentially by taking the complex conjugate bundles. By [24, Corollary 6.8.4], $SW_X(\bar{\mathfrak{s}}) = (-1)^{\chi_h(X)} SW_X(\mathfrak{s})$ so $B(X)$ is closed under the action of ± 1 on $H^2(X;\mathbb{Z})$.

Let $\widetilde{X} = X \# \bar{\mathbb{C}\mathbb{P}}^2$ be the blow-up of X. For every $n \in \mathbb{Z}$, there is a unique $\mathfrak{s}_n \in \mathrm{Spin}^c(\bar{\mathbb{C}\mathbb{P}}^2)$ with $c_1(\mathfrak{s}_n) = (2n+1)e^*$ where $e^* \in H^2(\widetilde{X};\mathbb{Z})$ is the Poincaré dual of the exceptional curve. By [28, §4.6.2], there is a bijection,

$$\mathrm{Spin}^c(X) \times \mathbb{Z} \to \mathrm{Spin}^c(\widetilde{X}), \quad (\mathfrak{s}_X, n) \mapsto \mathfrak{s}_X \# \mathfrak{s}_n,$$

given by a connected-sum construction with $c_1(\mathfrak{s}_X \# \mathfrak{s}_n) = c_1(\mathfrak{s}_X) + (2n+1)e^*$. Versions of the following have appeared in [14], [28, Theorem 4.6.7], and [18, Theorem 1].

Theorem 2.1. [18, Theorem 1] *Let X be a standard four-manifold and let $\widetilde{X} = X \# \bar{\mathbb{C}\mathbb{P}}^2$ be its blow-up. For all $\mathfrak{s} \in Spin^c(X)$ and $n \in \mathbb{Z}$,*

$$SW_{\widetilde{X}}(\mathfrak{s}\#\mathfrak{s}_n) = \begin{cases} SW_X(\mathfrak{s}) & \text{if } c_1^2(\mathfrak{s}\#\mathfrak{s}_n) \geq c_1^2(\widetilde{X}), \\ 0 & \text{if } c_1^2(\mathfrak{s}\#\mathfrak{s}_n) < c_1^2(\widetilde{X}). \end{cases}$$

Theorem 2.1 implies that for $\widetilde{X} = X \# \bar{\mathbb{C}\mathbb{P}}^2$

$$B(\widetilde{X}) = \{K + (2n+1)e^* : K \in B(X) \text{ and } K^2 - 4(n^2+n) \geq c_1^2(X)\}, \tag{2.4}$$

which gives the following:

Corollary 2.2. *A standard four-manifold X has SW-simple type if and only if its blow-up $\widetilde{X}$ has SW-simple type.*

2.2. **Donaldson invariants.**

2.2.1. *Definitions and the structure theorem.* We now recall the definition [21, §2] of the Donaldson series for standard four-manifolds. For any choice of $w \in H^2(X;\mathbb{Z})$, the Donaldson invariant is a linear function,

$$D_X^w : \mathbb{A}(X) \to \mathbb{R},$$

where $\mathbb{A}(X)$ is the symmetric algebra,

$$\mathbb{A}(X) = \mathrm{Sym}(H_{\mathrm{even}}(X;\mathbb{R})).$$

For $h \in H_2(X;\mathbb{R})$ and $x \in H_0(X;\mathbb{Z})$ a generator, we define $D_X^w(h^{\delta-2m}x^m) = 0$ unless

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod 4. \tag{2.5}$$

If (2.5) holds, then $D_X^w(h^{\delta-2m}x^m)$ is defined by pairing cohomology classes corresponding to elements of $\mathbb{A}(X)$ with the Uhlenbeck compactification of a moduli space of anti-self-dual SO(3) connections [1], [2], [17], [21].

A four-manifold has *KM-simple type* if for all $w \in H^2(X;\mathbb{Z})$ and all $z \in \mathbb{A}(X)$,

$$D_X^w(x^2 z) = 4D_X^w(z). \tag{2.6}$$

The Donaldson series is a formal power series,

$$\mathbf{D}_X^w(h) = D_X^w((1+\tfrac{1}{2}x)e^h), \quad h \in H_2(X;\mathbb{R}). \tag{2.7}$$

which determines all Donaldson invariants for standard manifolds of KM-simple type. The Donaldson series of a manifold with KM-simple type has the following description (see also [13, Theorems 5.9 & 5.13] for a proof by a different method):

Theorem 2.3. [21, Theorem 1.7 (a)] *Let X be a standard four-manifold with KM-simple type. Suppose that some Donaldson invariant of X is non-zero. Then there is a function,*

$$\beta_X : H^2(X;\mathbb{Z}) \to \mathbb{Q}, \tag{2.8}$$

such that $\beta_X(K) \neq 0$ for at least one and at most finitely many classes, K, which are integral lifts of $w_2(X) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ (the KM-basic classes*), and for any $w \in H^2(X;\mathbb{Z})$, one has the following equality of analytic functions of $h \in H_2(X;\mathbb{R})$:*

$$\mathbf{D}_X^w(h) = e^{Q_X(h)/2} \sum_{K \in H^2(X;\mathbb{Z})} (-1)^{(w^2+K\cdot w)/2}\beta_X(K)e^{\langle K,h\rangle}. \tag{2.9}$$

The following lemma reduces the proof of Conjecture 1.1 to proving that equation (1.2) holds.

Lemma 2.4. *Continue the assumptions of Theorem 2.3. If X satisfies* (1.2)*, then the KM basic classes and the SW basic classes coincide.*

Proof. The result follows from comparing equations (2.3) (which is equivalent to (1.2)) and (2.9) and from the linear independence of the functions $e^{r_i t}$ for different values of r_i. □

2.2.2. *Independence from w.* We now discuss the role of w. Proofs that the condition (2.6) is independent of w appear, in varying degrees of generality, in [21], [20], [31], [27]:

Theorem 2.5. [21], [27, Theorem 2] *Let X be a standard four-manifold. If* (2.6) *holds for one $w \in H^2(X;\mathbb{Z})$, then it holds for all w.*

The following allows us to work with a specific w:

Proposition 2.6. *If a standard four-manifold satisfies Witten's equality* (1.2) *for one $w \in H^2(X;\mathbb{Z})$, then X satisfies* (1.2) *for all $w \in H^2(X;\mathbb{Z})$.*

Proof. Assume that X satisfies (1.2) and hence (2.3) for $w_0 \in H^2(X;\mathbb{Z})$. Then, from the definition (2.1),

$$\begin{aligned} &e^{\frac{1}{2}Q_X(h)} \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w_0^2+K\cdot w_0)}\beta_X(K)e^{\langle K,h\rangle} \\ &\quad = 2^{2-(\chi_h-c_1^2)}e^{\frac{1}{2}Q_X(h)} \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w_0^2+K\cdot w_0)}SW'_X(K)e^{\langle K,h\rangle}. \end{aligned} \tag{2.10}$$

Let $B(X)=\{K_1,\dots,K_s\}$. Because Q_X is non-degenerate,

$$U = Q_X^{-1}(0) - \bigcup_{i<j} (K_i-K_j)^{-1}(0) \subset H_2(X;\mathbb{R})$$

is non-empty. If $r_i = \langle K_i, h\rangle$ for $h\in U$, then $r_i \neq r_j$ for $i\neq j$. For $t\in\mathbb{R}$ and $h\in U$ substituting th into (2.10) gives

$$\sum_{i=1}^{s}(-1)^{\frac{1}{2}(w_0^2+K_i\cdot w_0)}\beta_X(K_i)e^{r_it} = 2^{2-(\chi_h-c_1^2)}\sum_{i=1}^{s}(-1)^{\frac{1}{2}(w_0^2+K_i\cdot w_0)}SW'_X(K_i)e^{r_it}.$$

The preceding and the linear independence of the functions $e^{r_1t},\dots,e^{r_st}$ imply that

$$\beta_X(K) = 2^{2-(\chi_h-c_1^2)}SW'_X(K). \tag{2.11}$$

Let w be any other element of $H^2(X;\mathbb{Z})$. Theorem 2.5 implies that X has KM-simple type. The result then follows from equations (2.11) and (2.9). □

2.2.3. *Behavior under blow-ups.* We also note that (2.6) is invariant under blow-ups.

Proposition 2.7. *A standard four-manifold X has KM-simple type if and only if its blow-up $\widetilde{X}$ has KM-simple type.*

Proof. Assume $\widetilde{X}$ has KM-simple type. Then the blow-up formula $D^w_X(z) = D^w_{\widetilde{X}}(z)$ from [17, Theorem III.8.4] implies that

$$D^w_X(zx^2) = D^w_{\widetilde{X}}(zx^2) = 4D^w_{\widetilde{X}}(z) = D^w_X(z),$$

proving that X has KM-simple type. The opposite implication follows from [21, Proposition 1.9]. □

We also note the behavior of (1.2) under blow-up:

Theorem 2.8. [15, Theorem 8.9] *A standard four-manifold satisfies Witten's equality* (1.2) *if and only if its blow-up satisfies* (1.2).

2.2.4. *The Donaldson invariants determined by the conjecture.* Theorem 2.3 gives the following values for Donaldson invariants of manifolds satisfying Conjecture 1.2.

Lemma 2.9. *Let X be a standard four-manifold. Let $c(X) = \chi_h(X) - c_1^2(X)$. Then X satisfies equation* (1.2) *and has KM-simple type if and only if the Donaldson invariants of X satisfy $D^w_X(h^{\delta-2m}x^m) = 0$ when δ does not satisfy* (2.5) *and when δ satisfies* (2.5),

$$\begin{aligned} &D^w_X(h^{\delta-2m}x^m) \\ &\quad = \sum_{i+2k=\delta-2m} \sum_{K\in B(X)} (-1)^{\varepsilon(w,K)} \frac{SW'_X(K)(\delta-2m)!}{2^{k+c(X)-2-m}k!i!} \langle K,h\rangle^i Q_X(h)^k, \end{aligned} \tag{2.12}$$

where $\varepsilon(w,K) = \frac{1}{2}(w^2 + w\cdot K)$.

Proof. Assume that X satisfies equation (1.2) and hence (2.3) and has KM-simple type. By definition the Donaldson invariant will vanish unless δ satisfies (2.5). Then X satisfies equation (2.3) if and only if

$$\begin{aligned} &2^{c(X)-2}\sum_{d=0}^{\infty} \frac{1}{d!}D^w_X(h^d) + \frac{1}{d!}D^w_X(h^d x) \\ &= \left(\sum_{k=0}^{\infty}\frac{1}{2^k k!}Q_X(h)^k\right)\left(\sum_{i=0}^{\infty}\frac{1}{i!}\sum_{K\in B(X)}(-1)^{\varepsilon(w,K)}SW'_X(K)\langle K,h\rangle^i\right) \\ &= \sum_{d=0}^{\infty}\sum_{i+2k=d}\sum_{K\in B(X)}(-1)^{\varepsilon(w,K)}\frac{SW'_X(K)}{2^k k!i!}\langle K,h\rangle^i Q_X(h)^k. \end{aligned}$$

The parity restriction (1.2) implies that for $d \not\equiv -w^2 - 3\chi_h \pmod 2$,

$$D^w_X(h^d) + \tfrac{1}{2}D^w_X(h^d x) = 0,$$

while for $d \equiv -w^2 - 3\chi_h \pmod 2$, X satisfies (2.3) if and only if

$$\begin{aligned} &2^{c(X)-2}\left(D^w_X(h^d) + \tfrac{1}{2}D^w_X(h^d x)\right) \\ &\quad = \sum_{i+2k=d}\sum_{K\in B(X)}(-1)^{\varepsilon(w,K)}\frac{SW'_X(K)d!}{2^k k!i!}\langle K,h\rangle^i Q_X(h)^k. \end{aligned}$$

We can then read the value of $D^w_X(h^{\delta-2m}x^m)$ from the above as follows. If $\delta \equiv -w^2 - 3\chi_h \pmod 4$ and m is even, then $\delta - 2m \equiv -w^2 - 3\chi_h \pmod 4$ so by the KM-simple type condition (2.6) and the vanishing condition (2.5),

$$\begin{aligned} D^w_X(h^{\delta-2m}x^m) &= 2^m\left(D^w_X(h^{\delta-2m}) + \tfrac{1}{2}D^w_X(h^{\delta-2m}x)\right) \\ &= \sum_{i+2k=\delta-2m}\sum_{K\in B(X)}(-1)^{\varepsilon(w,K)}\frac{SW'_X(K)(\delta-2m)!}{2^{k+c(X)-2-m}k!i!}\langle K,h\rangle^i Q_X(h)^k. \end{aligned}$$

Similarly, if $\delta \equiv -w^2 - 3\chi_h \pmod 4$ and m is odd, then $\delta - 2m + 2 \equiv -w^2 - 3\chi_h \pmod 4$ so by the KM-simple type condition and the vanishing condition (2.5),

$$\begin{aligned} D^w_X(h^{\delta-2m}x^m) &= 2^{m-1} D^w_X(h^{\delta-2m}x) \\ &= 2^m \left(D^w_X(h^{\delta-2m}) + \tfrac{1}{2} D^w_X(h^{\delta-2m}x) \right) \\ &= \sum_{i+2k=\delta-2m} \sum_{K\in B(X)} (-1)^{\varepsilon(w,K)} \frac{SW'_X(K)(\delta-2m)!}{2^{k+c(X)-2-m}k!i!} \langle K,h\rangle^i Q_X(h)^k, \end{aligned}$$

as required.

Conversely, if the Donaldson invariants satisfy (2.12) then the KM-simple type condition (2.6) follows immediately. That X satisfies (1.2) follows from the arguments above. □

3. The SO(3) monopole cobordism formula

In this section, we review the SO(3) monopole cobordism formula. More detailed expositions appear in [8, 10, 11, 12, 7].

We will denote spin^c structures on X by $\mathfrak{s} = (W, \rho)$ where $W \to X$ is a rank-four, complex Hermitian vector bundle and ρ is a Clifford multiplication map. A spin^u structure $\mathfrak{t}$ is given by $\mathfrak{t} = (W \otimes E, \rho \otimes \mathrm{id}_E)$ where (W, ρ) is a spin^c structure and $E \to X$ is a rank-two complex Hermitian vector bundle. Such a spin^u structure $\mathfrak{t}$ defines an associated bundle $\mathfrak{g}_{\mathfrak{t}} = \mathfrak{su}(E)$ and characteristic classes

$$c_1(\mathfrak{t}) = c_1(W^+) + c_1(E), \quad \text{and} \quad p_1(\mathfrak{t}) = p_1(\mathfrak{g}_{\mathfrak{t}}).$$

We will often use the notation $\Lambda = c_1(\mathfrak{t})$, $\kappa = -\frac{1}{4}\langle p_1(\mathfrak{t}), [X]\rangle$, and $w = c_1(E)$ which is used to orient the moduli space. Let $\mathcal{M}_{\mathfrak{t}}$ denote the moduli space of SO(3) monopoles for the spin^u structure $\mathfrak{t}$ as defined in [10, Equation (2.33)]. The space $\mathcal{M}_{\mathfrak{t}}$ admits an S^1 action with fixed point subspaces given by M^w_κ, the moduli space of anti-self-dual connections on the bundle $\mathfrak{g}_{\mathfrak{t}}$, and by Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$ where $E = L_1 \oplus L_2$ and $\mathfrak{s} = W \otimes L_1$. For spin^c structures $\mathfrak{s}$ with $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}}$, we have $(c_1(\mathfrak{s}) - \Lambda)^2 = p_1(\mathfrak{t})$.

The dimension of M^w_κ is given by 2δ where

$$\delta = -p_1(\mathfrak{t}) - 3\chi_h.$$

The dimension of $\mathcal{M}_{\mathfrak{t}}$ is $2\delta + 2n_a(\mathfrak{t})$ where $n_a(\mathfrak{t})$ is the complex index of a Dirac operator defined by $\mathfrak{t}$ and $n_a = (I(\Lambda) - \delta)/4$ for

$$I(\Lambda) = \Lambda^2 - \frac{1}{4}(3\chi(X) + 7\sigma(X)) = \Lambda^2 + 5\chi_h(X) - c_1^2(X). \tag{3.1}$$

Thus, M^w_κ has positive codimension in $\mathcal{M}_{\mathfrak{t}}$ if and only if $I(\Lambda) > \delta$. Note also that because n_a is an integer, $I(\Lambda) \equiv \delta \pmod 4$ so

$$\Lambda^2 + c(X) \equiv \delta \pmod 4, \tag{3.2}$$

where we have used $I(\Lambda) = \Lambda^2 + c(X) + 4\chi_h(X)$ from (3.1).

The moduli space $\mathcal{M}_{\mathfrak{t}}$ is not compact but admits an Uhlenbeck-type compactification,

$$\bar{\mathcal{M}}_{\mathfrak{t}} \subset \cup_{\ell=0}^N \mathcal{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X),$$

where $\mathfrak{t}(\ell)$ is the spin^u structure satisfying $c_1(\mathfrak{t}(\ell)) = c_1(\mathfrak{t})$ and $p_1(\mathfrak{t}(\ell)) = p_1(\mathfrak{t}) + 4\ell$, [9, Theorem 4.20]. The S^1 action extends continuously over $\bar{\mathcal{M}}_{\mathfrak{t}}$. The closure of M^w_κ in $\bar{\mathcal{M}}_{\mathfrak{t}}$ is

the Uhlenbeck compactification $\bar{M}^w_\kappa$. There are additional fixed points of the S^1 action in $\bar{\mathcal{M}}_\mathfrak{t}$ of the form $M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)$. If $\bar{\mathbf{L}}^w_{\mathfrak{t},\kappa}$ and $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ are the links of $\bar{M}^w_\kappa$ and $M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)$ respectively in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$, then $\bar{\mathcal{M}}_\mathfrak{t}/S^1$ defines a compact, orientable cobordism between $\bar{\mathbf{L}}^w_{\mathfrak{t},\kappa}$ and the union, over $\mathfrak{s} \in \mathrm{Spin}^c(X)$, of the links $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$. If $I(\Lambda) > \delta$, then pairing certain cohomology classes with the link $\bar{\mathbf{L}}^w_{\mathfrak{t},\kappa}$ gives a multiple of the Donaldson invariant (see [11, Proposition 3.29]). As these cohomology classes are defined on the complement of the fixed point set in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$, the cobordism gives an equality between this multiple of the Donaldson invariant and the pairing of these cohomology classes with the union, over $\mathfrak{s} \in \mathrm{Spin}^c(X)$, of the links $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$. In [7], we computed a qualitative expression for this pairing, giving the following cobordism formula.

Theorem 3.1. [7] *Let X be a standard four-manifold of Seiberg-Witten simple type. Assume that $w, \Lambda \in H^2(X;\mathbb{Z})$ and $\delta, m \in \mathbb{N}$ satisfy*

1. $w - \Lambda \equiv w_2(X) \pmod 2$,
2. $I(\Lambda) > \delta$, *where $I(\Lambda)$ is defined in* (3.1),
3. $\delta \equiv -w^2 - 3\chi_h \pmod 4$,
4. $\delta - 2m \geq 0$.

Then for any $h \in H_2(X;\mathbb{R})$ and generator $x \in H_0(X;\mathbb{Z})$,

$$\begin{aligned} &D^w_X(h^{\delta-2m}x^m) \\ &\quad = \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2-\sigma)+\frac{1}{2}(w^2+(w-\Lambda)\cdot K)} SW'_X(K) f_{\delta,m}(\chi_h, c_1^2, K, \Lambda)(h), \end{aligned} \tag{3.3}$$

where for Q_X the intersection form of X, $c_1^2 = c_1^2(X)$, and $\chi_h = \chi_h(X)$ as defined in (1.1),

$$\begin{aligned} &f_{\delta,m}(\chi_h, c_1^2, K, \Lambda)(h) \\ &\quad = \sum_{i+j+2k=\delta-2m} a_{i,j,k}(\chi_h, c_1^2, K\cdot\Lambda, \Lambda^2, m)\langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X(h)^k, \end{aligned} \tag{3.4}$$

and the coefficients $a_{i,j,k}$ depend on i, j, k, χ_h, c_1^2, $c_1(\mathfrak{s})\cdot\Lambda$, Λ^2, and m.

Remark 3.2. The proof of Theorem 3.1 in [7] assumes that the gluing map constructed in [5] gives a continuous parametrization of a neighborhood of $M_\mathfrak{s} \times \Sigma$ in $\bar{\mathcal{M}}_\mathfrak{t}$ where $\Sigma \subset \mathrm{Sym}^\ell(X)$ is a smooth stratum. This gluing map is the analogue for SO(3) monopoles of the gluing map for the anti-self-dual equations constructed by Taubes in [30] (see also [2, §7.2]). We have established the existence of the map in [5] and expect that a proof of the continuity of this map with respect to Uhlenbeck limits should be similar to the proof in [4] of this property of the gluing map for the anti-self-dual equations. The remaining properties of this map assumed in [7] are that this map is injective and that it is surjective in the sense that elements of $\bar{\mathcal{M}}_\mathfrak{t}$ sufficiently close to $M_\mathfrak{s} \times \Sigma$ are in the map's image. The proof of these last two properties will require additional estimates on the derivative of the gluing map. Such results for the anti-self-dual equations have appeared in [2, §7.2.5, 7.2.6], [26], and [25] and for the Seiberg-Witten equations in [19] and [28, §4.5].

4. Determining the coefficients

In this section, we show how a four-manifold X satisfying Witten's Conjecture can determine the coefficients of the polynomial $f_{\delta,m}(\chi_h, c_1^2, c_1(\mathfrak{s}), \Lambda)$ appearing in equation (3.3) for $\chi_h = \chi_h(X)$ and $c_1^2 = c_1^2(X)$.

4.1. **Algebraic preliminaries.** We begin with a generalization of [17, Lemma VI.2.4] showing when equalities of the form (3.3) determine these coefficients.

Lemma 4.1. *Let V be a finite-dimensional real vector space. Let $T_1, \dots, T_n$ be linearly independent elements of the dual space V^*. Let Q be a quadratic form on V which is non-zero on $\cap_{i=1}^n \operatorname{Ker}(T_i)$. Then $T_1, \dots, T_n, Q$ are algebraically independent in the sense that if $F(z_0, \dots, z_n) \in \mathbb{R}[z_0, \dots, z_n]$ and $F(Q, T_1, \dots, T_n) : V \to \mathbb{R}$ is the zero map, then $F(z_0, \dots, z_n)$ is the zero element of $\mathbb{R}[z_0, \dots, z_n]$.*

Proof. We use induction on n. For $n = 1$, the result follows from [17, Lemma VI.2.4].

Assume that there is a polynomial $F(z_0, \dots, z_n)$ with $F(Q, T_1, \dots, T_n) : V \to \mathbb{R}$ the zero map. Assigning z_0 degree two and z_i degree one for $i > 0$, we can assume that F is homogeneous of degree d. Write $F(z_0, \dots, z_n) = z_n^r G(z_0, \dots, z_n)$ where z_n does not divide $G(z_0, \dots, z_n)$. Then, because $T_n^r G(Q, T_1, \dots, T_n)$ vanishes on V, $G(Q, T_1, \dots, T_n)$ must vanish on the dense set $T_n^{-1}(\mathbb{R}^*)$ and hence on V. Then write $G(z_0, \dots, z_n) = \sum_{i=0}^m G_i(z_0, \dots, z_{n-1}) z_n^{m-i}$. Because z_n does not divide $G(z_0, \dots, z_n)$, if $G(z_0, \dots, z_n)$ is not the zero polynomial, then $G_m(z_0, \dots, z_{n-1})$ is not zero. However, because $G(Q, T_1, \dots, T_n)$ is the zero function, $G_m(Q, T_1, \dots, T_{n-1})$ vanishes on $\operatorname{Ker}(T_n)$. If there are $c_1, \dots, c_{n-1} \in \mathbb{R}$ such that the restriction of $c_1T_1 + \dots c_{n-1}T_{n-1}$ to $\operatorname{Ker}(T_n)$ vanishes, then there is $c_n \in \mathbb{R}$ such that $c_1T_1 + \cdots + c_{n-1}T_{n-1} = c_nT_n$. Then, the linear independence of $T_1, \dots, T_n$ implies that $c_1 = \cdots = c_n = 0$. Hence, the restrictions of $T_1, \dots, T_{n-1}$ to $\operatorname{Ker}(T_n)$ are linearly independent. Induction then implies that $G_m(z_0, \dots, z_{n-1}) = 0$, a contradiction to $G(z_0, \dots, z_n)$ being non-zero. Hence, F must be the zero polynomial. □

Being closed under the action of ± 1, $B(X)$ is not linearly independent. Thus, to use Lemma 4.1 to determine the coefficients $a_{i,j,k}$ in (3.4) from examples of manifolds satisfying Witten's equality, we rewrite (2.12) and (3.3) as sums over a smaller set of spin^c structures.

Let $B'(X)$ be a fundamental domain for the ± 1 action on $B(X)$, so the projection map $B'(X) \to B(X)/\pm 1$ is a bijection. Lemma 2.9 has the following rephrasing.

Lemma 4.2. *Let X be a standard four-manifold. Then X satisfies equation* (1.2) *and has KM-simple type if and only if the Donaldson invariants of X satisfy $D_X^w(h^{\delta-2m}x^m) = 0$ for $\delta \not\equiv -w^2 - 3\chi_h \pmod 4$ and for $\delta \equiv -w^2 - 3\chi_h \pmod 4$ satisfy*

$$D_X^w(h^{\delta-2m}x^m) \tag{4.1}$$
$$= \sum_{i+2k=\delta-2m} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} n(K) \frac{SW_X'(K)(\delta-2m)!}{2^{k+c(X)-3-m} k! i!} \langle K, h \rangle^i Q_X(h)^k,$$

where $\varepsilon(w, K)$ is defined in Lemma 2.9 and

$$n(K) = \begin{cases} \frac{1}{2} & \text{if } K = 0, \\ 1 & \text{if } K \neq 0. \end{cases} \tag{4.2}$$

Proof. We will show that (2.12) holds if and only if (4.1) holds and so the lemma follows from Lemma 2.9.

Recall that $K \in B(X)$ if and only if $-K \in B(X)$. We rewrite the sum in (2.12) as a sum over $B'(X)$ by combining the K and $-K$ terms as follows. These two terms differ only in their factors of $(-1)^{\varepsilon(w,K)}$, $SW'_X(K)$, and $\langle K, h\rangle^i$. Because K is characteristic, we have

$$\tfrac{1}{2}[w^2 + w\cdot K] - \tfrac{1}{2}[w^2 - w\cdot K] \equiv w\cdot K \equiv w^2 \pmod 2.$$

From [24, Corollary 6.8.4], $SW'_X(-K) = (-1)^{\chi_h} SW'_X(K)$, so we can combine the distinct K and $-K$ terms in (2.12) using the equality

$$\begin{aligned}&(-1)^{\varepsilon(w,-K)} SW'_X(-K)\langle -K, h\rangle^i + (-1)^{\varepsilon(w,K)} SW'_X(K)\langle K, h\rangle^i \\ &\quad = \left((-1)^{\chi_h + w^2 + i} + 1\right)(-1)^{\varepsilon(w,K)} SW'_X(K)\langle K, h\rangle^i.\end{aligned} \tag{4.3}$$

In the sum (2.12), where $i + 2k = \delta - 2m$, we have $i \equiv \delta \pmod 2$. By (2.5), we have $\delta + w^2 \equiv \chi_h \pmod 4$ and so $\chi_h + w^2 + i \equiv \chi_h + w^2 + \delta \equiv 0 \pmod 2$. Thus, if $K \neq 0$ the K and $-K$ terms will combine as in (4.3) to give the factor of two in (4.1). When $K = 0$, the K and $-K$ terms are the same and so we must offset this factor of two with the given expression $n(K)$. □

We now perform a similar reduction for the sum in (3.3). Define

$$\begin{aligned}&b_{i,j,k}(\chi_h, c_1^2, K\cdot\Lambda, \Lambda^2, m) \\ &\quad = (-1)^{c(X)+i} a_{i,j,k}(\chi_h, c_1^2, -K\cdot\Lambda, \Lambda^2, m) + a_{i,j,k}(\chi_h, c_1^2, K\cdot\Lambda, \Lambda^2, m),\end{aligned} \tag{4.4}$$

where $a_{i,j,k}$ are the coefficients appearing in (3.4). By definition,

$$b_{i,j,k}(\chi_h, c_1^2, -K\cdot\Lambda, \Lambda^2, m) = (-1)^{c(X)+i} b_{i,j,k}(\chi_h, c_1^2, K\cdot\Lambda, \Lambda^2, m). \tag{4.5}$$

Lemma 4.3. *Continue the hypotheses of Theorem 3.1. Define*

$$\tilde\varepsilon(w, \Lambda, K) = \tfrac{1}{2}(w^2 - \sigma) + \tfrac{1}{2}(w^2 + (w - \Lambda)\cdot K), \tag{4.6}$$

and abbreviate the coefficient in (4.4) *by* $b_{i,j,k}(\Lambda, K) = b_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m)$. *Then,*

$$\begin{aligned}&D^w_X(h^{\delta-2m}x^m) \\ &\quad = \sum_{i+j+2k=\delta-2m}\ \sum_{K\in B'(X)} n(K)(-1)^{\tilde\varepsilon(w,\Lambda,K)} SW'_X(K) b_{i,j,k}(\Lambda, K)\langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X(h)^k,\end{aligned} \tag{4.7}$$

where $n(K)$ is defined by (4.2).

Proof. Because $w - \Lambda$ is characteristic and because $K^2 = c_1^2(X)$,

$$\tilde\varepsilon(w, \Lambda, -K) = \tilde\varepsilon(w, \Lambda, K) - (w - \Lambda)\cdot K) \equiv \tilde\varepsilon(w, \Lambda, K) + c_1^2(X) \pmod 2.$$

For $K \neq 0$, we can combine the K and $-K$ terms in the sum (3.3) as in (4.3) to obtain the expression $b_{i,j,k}$ in (4.4). For $K = 0$, the factor of $n(K) = 1/2$ is necessary because there is only one such term in the sum, instead of the two identical terms given by (4.4). □

4.2. **The example manifolds.** A manifold with the following properties can be used with Lemmas 4.1, 4.2, and 4.3 to determine many of the coefficients $b_{i,j,k}$ in (4.7).

Definition 4.4. A standard four-manifold is *useful* if

(1) X has SW-simple type, and $|B'(X)| = 1$,
(2) X satisfies Witten's equality (4.1),
(3) There are $f_1, f_2 \in B(X)^\perp$ with $f_i^2 = 0$ and $f_1 \cdot f_2 = 1$ such that $\{f_1, f_2\} \cup B'(X)$ is linearly independent,
(4) If f_1, f_2 are the cohomology classes in the previous condition, then the restriction of Q_X to $\left(\cap_{i=1}^2 \operatorname{Ker}(f_i)\right) \cap \left(\cap_{K \in B'(X)} \operatorname{Ker}(K)\right)$ is non-zero.

We show the existence of a family of useful four-manifolds in the following.

Lemma 4.5. *For every $h = 2, 3, \ldots$, there is a useful four-manifold X_h with $\chi_h(X_h) = h$, $c_1^2(X) = h - 3$, and $c(X) = 3$.*

Proof. In [16], Fintushel, J. Park, and Stern construct examples of standard four-manifolds X_p and X'_p for $p \in \mathbb{Z}$ with $p \geq 4$ with $c_1^2(X_p) = 2p - 7$ and $c_1^2(X'_p) = 2p - 8$ and both satisfying $c_1^2 = \chi_h - 3$. In addition, $|B(X_p)/\pm 1| = |B(X'_p)/\pm 1| = 1$. The manifolds from [16] define a ray in the (χ_h, c_1^2) plane but the restriction $p \geq 4$ implies that they do not fill in the point $\chi_h = 2$, $c_1^2 = -1$. To fill this hole, define $X_3 = K3 \# \bar{\mathbb{C}}\mathbb{P}^2$.

As shown following Lemma 3.4 in [16], for $p \geq 4$, X_p and X'_p are rational blow-downs of the elliptic surfaces $E(2p-4)$ and $E(2p-5)$ respectively along taut configurations (in the sense of [15, §7]) of embedded spheres. These elliptic surfaces have SW-simple type and satisfy Conjecture 1.1 (see, for example, [15, Theorem 8.7]). By [15, Theorem 8.9], these properties are preserved under rational blowdown and hence also hold for X_p and X'_p when $p \geq 4$. For $X_3 = K3 \# \bar{\mathbb{C}}\mathbb{P}^2$, these properties hold because they hold for $K3 = E(2)$, by [21], and because these properties are preserved under blow-ups by Theorem 2.8.

Recall that a four-manifold X is abundant if there are $f_1, f_2 \in B(X)^\perp$ with $f_i^2 = 0$ and $f_1 \cdot f_2 = 1$. By [10, Corollary A.3], if X is simply connected and the SW basic classes are all multiples of a single cohomology class, then X is abundant. This together with the number of SW basic classes for these manifolds implies that X_p and X'_p are abundant for $p \geq 4$. The definition of $X_3 = K3 \# \bar{\mathbb{C}}\mathbb{P}^2$, the computation of $B(K3) = \{0\}$ (see, for example, [15, Equation (17)]), the blow-up formula (2.4), imply that $B(X_3)^\perp$ contains $H^2(K3)$ and thus X_3 is abundant.

To show the linear independence property, we first claim that $0 \notin B(X)$ where $X = X_p$ or $X = X'_p$. If $0 \in B(X)$ and X has SW-simple type, then $c_1^2(X) = 0$. Hence, we only have to consider the case X'_4. From [16], X'_4 is a rational blow-down of the elliptic surface $E(3)$ along the configuration of curves labeled C_1 in [16, Theorem 1.2]. By [16, Theorem 1.2], the basic classes of X'_4 are in one-to-one correspondence with $K \in B(E(3))$ satisfying $K \cdot S_0 = \pm 1$ where S_0 is the Poincaré dual of a curve in the configuration C_1. Because there are two such elements of $B(E(3))$, we see $|B(X'_4))| = 2$ so $0 \notin B(X'_4)$. If $f_1, f_2 \in B(X_p)^\perp$ are as in the definition of usefulness, if $K \in B(X_p)$, and if $af_1 + bf_2 + cK = 0$ for $a, b, c \in \mathbb{R}$, then we have $a = f_2 \cdot (af_1 + bf_2 + cK) = 0$, $b = f_1 \cdot (af_1 + bf_2 + cK) = 0$, so $cK = 0$. Because $K \neq 0$, we see that $c = 0$ and so $\{f_1, f_2, K\}$ is linearly independent.

Finally, the condition on Q_X in the definition of usefulness holds by observing that $b^+ \geq 3$ for all the example manifolds and $\{f_1, f_2\}$ only span a one-dimensional positive definite subspace. □

4.3. **The blow-up formulas.** To determine the coefficients $b_{i,j,k}$ for a sufficiently wide range of values of χ_h, c_1^2, Λ^2, and $\Lambda \cdot K$, we will need to work on the blow-up of the useful four-manifolds of Lemma 4.5. Thus, let $\widetilde{X}(n)$ be the blow-up of X at n points where X is one of the useful four-manifolds discussed in Lemma 4.5. For $0 \leq m \leq n$, we will consider $H^2(\widetilde{X}(m);\mathbb{Z})$ as a subspace of $H^2(\widetilde{X}(n);\mathbb{Z})$ using the inclusion defined by the pullback of the blowdown map. Let $e_1 \ldots, e_n \in H_2(\widetilde{X}(n);\mathbb{Z})$ be the homology classes of the exceptional curves and let $e_u^* = \mathrm{PD}[e_u]$.

To describe $B(\widetilde{X}(n))$, we introduce the following notation. Let $\pi_u : (\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$ be projection onto the u-th factor. For $K \in B(X)$ and $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$, define

$$K_\varphi = K + \sum_{u=1}^n (-1)^{\pi_u(\varphi)} e_u^*,$$

and abbreviate $K_0 = K + \sum_{u=1}^n e_u^*$. Then, by (2.4), if $0 \notin B(X)$,

$$B'(\widetilde{X}(n)) = \{K_\varphi : K \in B'(X) \text{ and } \varphi \in (\mathbb{Z}/2\mathbb{Z})^n\}.$$

Even if $B'(X)$ is linearly independent, $B'(\widetilde{X}(n))$ will not be for $n \geq 2$. To rewrite Lemma 4.3 in terms of linearly independent classes, we will require the following combinatorial result.

Lemma 4.6. *For $f : \mathbb{Z} \to \mathbb{R}$ and $p, q \in \mathbb{Z}$, define $(\nabla_p^q f)(x) = f(x) + (-1)^q f(x+p)$. For $a \in \mathbb{Z}/2\mathbb{Z}$ and $p \in \mathbb{Z}$, define $pa = -\frac{1}{2}(-1 + (-1)^a)p$. Then,*

$$\sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n q_u \pi_u(\varphi)} f(x + \sum_{u=1}^n p_u \pi_u(\varphi)) = (\nabla_{p_1}^{q_1} \nabla_{p_2}^{q_2} \ldots \nabla_{p_n}^{q_n} f)(x).$$

and for C a constant,

$$(\nabla_{p_n}^{q_n} \nabla_{p_{n-1}}^{q_{n-1}} \ldots \nabla_{p_1}^{q_1} C) = \begin{cases} 0 & \text{if there is } u \text{ with } 1 \leq u \leq n \text{ and } q_u \equiv 1 \pmod 2, \\ 2^n C & \text{if for all } u \text{ with } 1 \leq u \leq n,\ q_u \equiv 0 \pmod 2. \end{cases} \tag{4.8}$$

Proof. The proof uses induction on n. For $n = 1$, the statement is trivial. Define,

$$(L_{p_1,\ldots,p_n}^{q_1,\ldots,q_n} f)(x) = \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n q_u \pi_u(\varphi)} f(x + \sum_{u=1}^n p_u \pi_u(\varphi)).$$

For $n \geq 2$, the preceding expression can be expanded as

$$\begin{aligned}
&\sum_{\varphi\in\pi_n^{-1}(0)} (-1)^{\sum_{u=1}^{n-1} q_u\pi_u(\varphi)} f(x+\sum_{u=1}^{n-1} p_u\pi_u(\varphi)) \\
&\quad + (-1)^{q_n} \sum_{\varphi\in\pi_n^{-1}(1)} (-1)^{\sum_{u=1}^{n-1} q_u\pi_u(\varphi)} f(x+p_n+\sum_{u=1}^{n-1} p_u\pi_u(\varphi)) \\
&= (L^{q_1,\dots,q_{n-1}}_{p_1,\dots,p_{n-1}} f)(x) + (-1)^{q_n}(L^{q_1,\dots,q_{n-1}}_{p_1,\dots,p_{n-1}} f)(x+p_n) \\
&= (\nabla^{q_n}_{p_n}(L^{q_1,\dots,q_{n-1}}_{p_1,\dots,p_{n-1}} f))(x),
\end{aligned}$$

where in the penultimate step we have identified $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ with $\pi_n^{-1}(0)$ and with $\pi_n^{-1}(1)$ as sets. The first statement of the lemma then follows by induction.

Equation (4.8) follows from the equality,

$$\nabla^q_p C = C + (-1)^q C = \begin{cases} 0 & \text{if } q\equiv 1 \pmod 2, \\ 2C & \text{if } q\equiv 0 \pmod 2, \end{cases}$$

and induction on n. □

We now rewrite Lemmas 4.2 and 4.3 in terms of linearly independent classes.

Lemma 4.7. *Continue the notation of the preceding paragraphs and Definition 4.4 and let X be a useful four-manifold. For $w\in H^2(X;\mathbb{Z})\subset H^2(\widetilde{X}(n);\mathbb{Z})$, let $\tilde w = w+\sum_{u=1}^n w_u e_u^*$. Let $\Lambda\in H^2(\widetilde{X}(n);\mathbb{Z})$ satisfy $I(\Lambda)>\delta$ and $\Lambda-\tilde w\equiv w_2(\widetilde{X}(n)) \pmod 2$. Define $b_{i,j,k}(\Lambda^2,\Lambda\cdot K_\varphi) = b_{i,j,k}(\chi_h(\widetilde{X}(n)), c_1^2(\widetilde{X}(n)), K_\varphi\cdot\Lambda, \Lambda^2, m)$. Then,*

$$\begin{aligned}
&(-1)^{\varepsilon(\tilde w,K_0)} \sum_{i_0+\cdots+i_n+2k=\delta-2m} \binom{i_0+\cdots+i_n}{i_0,\ i_1,\ \dots,i_n} \frac{SW'_X(K)(\delta-2m)!}{2^{k+c+n-3-m}k!i!} \\
&\qquad \times p^w(i_1,i_2,\dots,i_n)\langle K,h\rangle^{i_0}\prod_{u=1}^n \langle e_u^*,h\rangle^{i_u} Q_X(h)^k \\
&= (-1)^{\tilde\varepsilon(\tilde w,\Lambda,K_0)} \sum_{i_0+\cdots+i_n+j+2k=\delta-2m} \binom{i_0+\cdots+i_n}{i_0,i_1,\ \dots,i_n} SW'_X(K) \\
&\times \sum_{\varphi\in(\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n (1+i_u)\pi_u(\varphi)} b_{i,j,k}(\Lambda^2,\Lambda\cdot K_\varphi)\langle K,h\rangle^{i_0}\prod_{u=1}^n\langle e_u^*,h\rangle^{i_u}\langle\Lambda,h\rangle^j Q_X(h)^k,
\end{aligned} \tag{4.9}$$

where $c=c(X)$ and

$$p^w(i_1,i_2,\dots,i_n) = \begin{cases} 0 & \text{if there is } q \text{ with } 1\le q\le n \text{ and } w_q+i_q\equiv 1 \pmod 2, \\ 2^n & \text{if for all } q \text{ with } 1\le q\le n,\ w_q+i_q\equiv 0 \pmod 2. \end{cases} \tag{4.10}$$

Proof. Comparing (4.1) and (4.7) yields, for $\varepsilon(\tilde{w},\varphi)=\frac{1}{2}(\tilde{w}^2+\tilde{w}\cdot K_\varphi)$

$$\begin{aligned}&\sum_{i+2k=\delta-2m}\sum_{\varphi\in(\mathbb{Z}/2\mathbb{Z})^n}(-1)^{\varepsilon(\tilde{w},\varphi)}\frac{SW'_X(K)(\delta-2m)!}{2^{k+c+n-3-m}k!i!}\langle K_\varphi,h\rangle^i Q_X(h)^k\\ &=\sum_{i+j+2k=\delta-2m}\sum_{\varphi\in(\mathbb{Z}/2\mathbb{Z})^n}(-1)^{\tilde{\varepsilon}(w,\Lambda,K_\varphi)}SW'_X(K)b_{i,j,k}(\Lambda,r)\langle K_\varphi,h\rangle^i\langle\Lambda,h\rangle^j Q_X(h)^k.\end{aligned}\tag{4.11}$$

For $\varphi\in(\mathbb{Z}/2\mathbb{Z})^n$,

$$\varepsilon(\tilde{w},\varphi)\equiv\tfrac{1}{2}(\tilde{w}^2+\tilde{w}\cdot K_\varphi)\equiv\tfrac{1}{2}(\tilde{w}^2+\tilde{w}\cdot K_0)+\sum_{u=1}^n w_u\pi_u(\varphi)\pmod 2.\tag{4.12}$$

By the multinomial theorem, for $\varphi\in(\mathbb{Z}/2\mathbb{Z})^n$ we can expand the factor $\langle K_\varphi,h\rangle^i$ as

$$\begin{aligned}&\langle K_\varphi,h\rangle^i\\ &=\sum_{i_0+\cdots+i_n=i}\binom{i}{i_0,\ i_1,\ \dots,i_n}(-1)^{\sum_{u=1}^n\pi_u(\varphi)i_u}\langle K,h\rangle^{i_0}\prod_{u=1}^n\langle e_u^*,h\rangle^{i_u},\end{aligned}\tag{4.13}$$

where, for $i=i_0+\cdots+i_n$,

$$\binom{i}{i_0,\ i_1,\ \dots,i_n}=\frac{i!}{i_0!i_1!\dots i_n!}.$$

The equalities (4.12) and (4.13) imply that we can rewrite the left-hand-side of (4.11) as

$$\begin{aligned}&\sum_{i+2k=\delta-2m}\sum_{\varphi\in(\mathbb{Z}/2\mathbb{Z})^n}(-1)^{\varepsilon(\tilde{w},\varphi)}\frac{SW'_X(K)(\delta-2m)!}{2^{k+c+n-3-m}k!i!}\langle K_\varphi,h\rangle^i Q_X(h)^k\\ &=(-1)^{\varepsilon(\tilde{w},K_0)}\sum_{i_0+\cdots+i_n+2k=\delta-2m}\binom{i_0+\cdots+i_n}{i_0,\ \dots,i_n}\frac{SW'_X(K)(\delta-2m)!}{2^{k+c+n-3-m}k!i!}\\ &\quad\times\sum_{\varphi\in(\mathbb{Z}/2\mathbb{Z})^n}(-1)^{\sum_{u=1}^n\pi_u(\varphi)(w_u+i_u)}\langle K,h\rangle^{i_0}\prod_{u=1}^n\langle e_u^*,h\rangle^{i_u}Q_X(h)^k.\end{aligned}\tag{4.14}$$

Applying Lemma 4.6, we write the sum over $\varphi\in(\mathbb{Z}/2\mathbb{Z})^n$ in (4.14) as

$$\sum_{\varphi\in(\mathbb{Z}/2\mathbb{Z})^n}(-1)^{\sum_{u=1}^n\pi_u(\varphi)(w_u+i_u)}=\nabla_0^{w_1+i_1}\dots\nabla_0^{w_n+i_n}1.$$

Equation (4.8) shows that the preceding expression equals $p^w(i_1,\dots,i_n)$ as defined in (4.10). Therefore (4.14) shows that the left-hand-side of (4.11) equals the left-hand-side of (4.9).

We now rewrite the right-hand-side of (4.11). The discussion is essentially the same as that for the left-hand-side. However, note that

$$\tilde{\varepsilon}(\tilde{w},\Lambda,K_\varphi)-\tilde{\varepsilon}(\tilde{w},\Lambda,K_0)=\tfrac{1}{2}(\Lambda-\tilde{w})\cdot(K_\varphi-K_0).$$

Because

$$K_\varphi-K_0=\sum_{u=1}^n((-1)^{\pi_u(\varphi)}-1)e_u^*=-2\sum_{u=1}^n\pi_u(\varphi)e_u^*,$$

and because $\Lambda-\tilde{w}$ is characteristic,

$$\tfrac{1}{2}(\Lambda-\tilde{w})\cdot(K_\varphi-K_0)\equiv\sum_{u=1}^{n}\pi_u(\varphi)\pmod 2.$$

This replaces the orientation change computed in (4.12), showing that the right-hand-side of (4.11) equals the right-hand-side of (4.9). □

4.4. **Determining the coefficients** $b_{i,j,k}$**.** We now apply Lemmas 4.1 and 4.7 to the manifolds discussed in Lemma 4.5 to determine many of the coefficients $b_{i,j,k}$.

Proposition 4.8. *For any integers x,y and for any integers $m\geq 0$, $n>0$, and $\chi_h\geq 2$ and for any non-negative integers i, j, k satisfying $i+j+2k=\delta-2m$, $i\geq n$, and $2y>\delta-4\chi_h-3-n$, the coefficients $b_{i,j,k}(\chi_h,c_1^2,\Lambda\cdot K,\Lambda^2,m)$ in (4.4) satisfy*

$$b_{i,j,k}(\chi_h,\chi_h-3-n,2x,2y,m)=\begin{cases}(-1)^{x+y}\frac{(\delta-2m)!}{k!i!}2^{m-k-n} & \text{if } j=0,\\ 0 & \text{if } j>0.\end{cases}$$

Proof. For X one of the useful four-manifolds discussed in Lemma 4.5, let $\widetilde{X}(n)$ be the blow-up of X at n points. We apply Lemma 4.7 with Λ given by

$$\Lambda=(y+2x^2)f_1+f_2+2xe_1^*,$$

where $f_1,f_2\in B(X)^\perp$ are the cohomology classes of Definition 4.4 satisfying $f_i^2=0$ and $f_1\cdot f_2=1$. Then,

$$\Lambda^2=2y,\quad \Lambda\cdot K_0=-2x.$$

The condition $2y>\delta-4\chi_h-3-n$ implies that $I(\Lambda)>\delta$. Observe that

$$(K_\varphi-K_0)\cdot\Lambda=\begin{cases}0 & \text{if } \pi_1(\varphi)=0,\\ 4x & \text{if } \pi_1(\varphi)=1.\end{cases}$$

If we write $\tilde{w}=w+\sum_{u=1}^{n}w_ue_u^*$, then the requirement that $\Lambda-\tilde{w}$ is characteristic implies that $w_u\equiv 1\pmod 2$ for all u. Then the coefficient of the term,

$$K^{i_0}(e_1^*)^{i_1}\dots(e_n^*)^{i_n}\Lambda^jQ_X^k, \tag{4.15}$$

on the left-hand-side of (4.9) will vanish if $j>0$ and for $j=0$ equals

$$(-1)^{\varepsilon(\tilde{w},K_0)}\binom{i}{i_0,i_1,\dots,i_n}\frac{SW_X'(K)(\delta-2m)!}{2^{k+n-m}k!i!}p^w(i_i,\dots,i_n), \tag{4.16}$$

where $i=i_0+\cdots+i_n$.

The coefficient of the term (4.15) on the right-hand-side of (4.9) is

(4.17)

$$(-1)^{\tilde{\varepsilon}(\tilde{w},\Lambda,K_0)}\binom{i}{i_0,\dots,i_n}SW_X'(K)\left(b_{i,j,k}(\Lambda^2,-2x)\left(\sum_{\varphi\in\pi_1^{-1}(0)}(-1)^{\sum_{u=1}^{n}(1+i_u)\pi_u(\varphi)}\right)\right.$$
$$\left.+b_{i,j,k}(\Lambda^2,2x)\left(\sum_{\varphi\in\pi_1^{-1}(1)}(-1)^{\sum_{u=1}^{n}(1+i_u)\pi_u(\varphi)}\right)\right).$$

Equation (4.8) shows that for $a = 0, 1$,

$$\sum_{\varphi\in\pi^{-1}(a)} (-1)^{\sum_{u=2}^{n}(1+i_u)\pi_u(\varphi)} = \begin{cases} 0 & \text{if there is } q,\ 2 \le q \le n, \text{ with } i_q \equiv 0 \pmod 2, \\ 2^{n-1} & \text{if } i_q \equiv 1 \pmod 2 \text{ for all } q \text{ with } 2 \le q \le n. \end{cases}$$

We write $p^1(i_2, \dots, i_n)$ for the preceding expression. Hence,

$$\begin{aligned} \sum_{\varphi\in\pi_1^{-1}(0)} (-1)^{\sum_{u=1}^{n}(1+i_u)\pi_u(\varphi)} &= p^1(i_2, \dots, i_n), \\ \sum_{\varphi\in\pi_1^{-1}(1)} (-1)^{\sum_{u=1}^{n}(1+i_u)\pi_u(\varphi)} &= (-1)^{1+i_1} p^1(i_2, \dots, i_n). \end{aligned}$$

The equality (4.5), the equality $\Lambda^2 - \delta \equiv c(\widetilde{X}(n)) \pmod 4$ from (3.2), our assumption that $\Lambda^2 \equiv 0 \pmod 2$, and $\delta \equiv i + j \pmod 2$ imply that

$$b_{i,j,k}(\Lambda^2, -2x) = (-1)^{\delta+i} b_{i,j,k}(\Lambda^2, 2x) = (-1)^j b_{i,j,k}(\Lambda^2, 2x).$$

Because $\Lambda - \tilde{w}$ is characteristic, we have $(\Lambda - \tilde{w})^2 \equiv \sigma \pmod 8$ and $\Lambda^2 \equiv \Lambda\cdot(\Lambda - \tilde{w}) \pmod 2$ so $\Lambda \cdot \tilde{w} \equiv 0 \pmod 2$. Thus, $(\Lambda - \tilde{w})^2 \equiv \sigma \pmod 8$ implies that $\Lambda^2 + \tilde{w}^2 \equiv \sigma \pmod 4$ and so $\frac{1}{2}(\tilde{w}^2 - \sigma) \equiv \frac{1}{2}\Lambda^2 \pmod 2$. Then, by the definitions of $\varepsilon(\tilde{w}, K_0)$ and $\tilde{\varepsilon}(\tilde{w}, \Lambda, K_0)$,

$$\tilde{\varepsilon}(\tilde{w}, \Lambda, K_0) - \varepsilon(\tilde{w}, K_0) = \tfrac{1}{2}(w^2 - \sigma) - \tfrac{1}{2}\Lambda \cdot K_0 \equiv \tfrac{1}{2}(\Lambda^2 + \Lambda \cdot K_0) \pmod 2.$$

From the preceding, we can rewrite (4.17) as

$$(-1)^{\varepsilon(\tilde{w},K_0)+x+y} \binom{i}{i_0, \dots, i_n} SW'_X(K) b_{i,j,k}(\Lambda^2, 2x) p^1(i_2, \dots, i_n) \left((-1)^j - (-1)^{i_1}\right). \tag{4.18}$$

Lemma 4.1 implies that the coefficients (4.16) and (4.18) must be equal. For this to be a non-trivial relation, we must have $p^1(i_2, \dots, i_n)$ non-zero and thus we must have $i_u \equiv 1 \pmod 2$ for $u = 2, \dots, n$. For j even, take $i_1 = \dots = i_n = 1$ and $i_0 = i - n$ while for j odd take $i_1 = 0$, $i_2 = \dots = i_n = 1$, and $i_0 = i - n + 1$ to get the desired equalities. □

Remark 4.9. Proposition 4.8 only determines the coefficients $b_{i,j,k}(\chi_h, c_1^2, \Lambda\cdot K, \Lambda^2, m)$ for $i \ge \chi_h - c_1^2 - 3$. An earlier version of this paper failed to note that because $p^1(i_2, \dots, i_n)$ vanishes for low values of i, the resulting relations were trivial and gave no information about $b_{i,j,k}$.

Remark 4.10. We now describe some limitations on the ability of the equality (4.9) to determine the coefficients $b_{i,j,k}$. For χ_h, c_1^2, Λ^2, and m fixed, define $c_{i,j,k} : \mathbb{Z} \to \mathbb{R}$ by $c_{i,j,k}(x) = b_{i,j,k}(\chi_h, c_1^2, x, \Lambda^2, m)$. If, in the notation of Proposition 4.8, one takes

$$\Lambda = y f_1 + f_2 + \sum_{u=1}^{n} \lambda_u e_u^*,$$

then Lemma 4.6 implies that the coefficient of the term (4.15) on the right-hand-side of (4.9) would be

$$\nabla_{2\lambda_1}^{i_1+1} \dots \nabla_{2\lambda_n}^{i_n+1} c_{i,j,k}(\Lambda \cdot K_0).$$

Because $\nabla_{2\lambda_1}^{1} \dots \nabla_{2\lambda_n}^{1} p(x) = 0$ for any polynomial $p(x)$ of degree $n-1$ or less, the arguments of Proposition 4.8 cannot determine the coefficients $b_{0,j,k}$. Arguing by induction on $i = v$

and varying $i_1, \dots, i_v$, one can show that the arguments of Proposition 4.8 determine $b_{i,j,k}$ only up to a polynomial of degree $n-i-1$ in $\Lambda \cdot K$.

Proof of Theorem 1.2 for four-manifolds with $c_1^2 \geq \chi_h - 3$. Assume that Y is a standard four-manifold with $c_1^2(Y) \geq \chi_h(Y) - 3$. Let X_h be the useful four-manifold of Lemma 4.5 with $\chi_h(X_h) = \chi_h(Y)$. By Theorem 2.8 and by blowing-up Y if necessary, we can assume that $c_1^2(Y) = c_1^2(X_h)$. Let $\widetilde{Y}$ and $\widetilde{X}_h$ be the blow-ups of Y and X_h respectively at a point. Let $e^* \in H^2(\widetilde{Y})$ be the Poincare dual of the exceptional curve. For $w \in H^2(Y)$ characteristic, define $\tilde{w} = w + e^* \in H^2(\widetilde{Y})$. For $B'(Y) = \{K_1, \dots, K_b\}$, there are $f_1, f_2 \in H^2(Y)$ with $K_1 \cdot f_i = 0$, $f_i^2 = 0$, and $f_1 \cdot f_2 = 1$ by [10, Corollary A.3]. For a given δ, we can pick an integer a such that for $\Lambda = 2(af_1 + f_2) \in H^2(Y) \subset H^2(\widetilde{Y})$, we have $\Lambda^2 = 8a$ and $I(\Lambda) > \delta$. Because $I(\Lambda) > \delta$ and $\Lambda - \tilde{w}$ is characteristic, we can use this $\tilde{w}$ and Λ in Lemma 4.3 to compute the degree δ Donaldson invariant of Y. Because $\Lambda^2 \equiv 0 \pmod 2$ and K_i is characteristic, $\Lambda \cdot K_i \equiv 0 \pmod 2$ for all $K_i \in B(\widetilde{Y})$. Proposition 4.8 then only gives an expression for the coefficients

$$b_{i,j,k}(\Lambda, K_i \pm e^*) = b_{i,j,k}(\chi_h(\tilde{Y}), c_1^2(\tilde{Y}), \Lambda \cdot (K_i \pm e^*), 8a, m)$$

appearing in (4.7) for $i \geq 1$. We first show that we can ignore the terms in (4.7) with $i = 0$.

Because $\tilde{w} - \Lambda$ is characteristic,

$$\begin{aligned} \tilde{\varepsilon}(\tilde{w}, \Lambda, K_i + e^*) &\equiv \tilde{\varepsilon}(\tilde{w}, \Lambda, K_i - e^*) + (\tilde{w} - \Lambda) \cdot e^* \pmod 2 \\ &\equiv \tilde{\varepsilon}(\tilde{w}, \Lambda, K_i - e^*) + 1 \pmod 2. \end{aligned}$$

Because $\Lambda \cdot (K_i + e^*) = \Lambda \cdot (K_i - e^*)$,

$$b_{i,j,k}(\Lambda, K_i + e^*) = b_{i,j,k}(\Lambda, K_i - e^*).$$

Finally, because $n(K_i \pm e^*) = 1$, the terms for $K_i + e^*$ and $K_i - e^*$ in (4.7) with $i = 0$ will cancel out. Thus, we may ignore the $i = 0$ terms.

Because $\widetilde{w}$ is characteristic, the definition of $\tilde{\varepsilon}$ in (4.6) implies that

$$\tilde{\varepsilon}(\tilde{w}, \Lambda, K_i \pm e^*) + \tfrac{1}{2}\Lambda \cdot (K_i \pm e^*) \equiv \varepsilon(\tilde{w}, K_i \pm e^*) \pmod 2.$$

Then, the computation of the coefficients $b_{i,j,k}$ in Proposition 4.8 and the vanishing of the terms with $i = 0$ allows us to rewrite (4.7) as

$$\begin{aligned} &D^{\tilde{w}}_{\widetilde{Y}}(h^{\delta-2m}x^m) \\ &= \sum_{i+j+2k=\delta-2m} \sum_{K \in B'(\widetilde{Y})} (-1)^{\varepsilon(\tilde{w},K)} SW'_X(K) \frac{(\delta-2m)!}{k!i!} 2^{m-k-1} \langle K, h\rangle^i Q_X(h)^k. \end{aligned} \tag{4.19}$$

Comparing (4.19) with (4.1), noting that $c(\widetilde{Y}) = 4$, and applying Lemma 4.2 then shows that Witten's conjecture holds for $\widetilde{Y}$ and thus for Y. □

Proof of Theorem 1.2 for abundant four-manifolds. We now show how Proposition 4.8 suffice to prove the Witten conjecture for abundant manifolds. For $w \in H^2(Y;\mathbb{Z})$ and $h \in H_2(Y;\mathbb{R})$, we define

$$SW^w_{Y,i}(h) = \sum_{K \in B(Y)} (-1)^{\varepsilon(w,K)} SW'_Y(K) \langle K, h\rangle^i.$$

We will use the following vanishing result for abundant four-manifolds.

Theorem 4.11. [3] *Theorem 3.1 implies that if Y is a standard and abundant four-manifold and w is characteristic, then $SW^w_{Y,i}$ vanishes for $i < c(Y) - 2$.*

By the argument from Lemma 4.2, for w characteristic (so $w^2 \equiv c_1^2(Y) \pmod 2$),

$$SW^w_{Y,i}(h) = (1 + (-1)^{c(X)+i}) \sum_{K \in B'(Y)} (-1)^{\varepsilon(w,K)} n(K) SW'_Y(K) \langle K, h\rangle^i. \tag{4.20}$$

By Theorem 2.8, it suffices to prove that Conjecture 1.1 holds for the blow-up of Y at any number of points. We can therefore assume that $c_1^2(Y) = \chi_h(Y) - 3 - n$ for $n \geq 1$. For any non-negative integers δ and m satisfying $\delta - 2m \geq 0$, pick an integer a such that $8a > \delta - 5\chi_h(Y) - c_1^2(Y)$. Let $f_1, f_2 \in B(Y)^\perp$ satisfy $f_1 \cdot f_2 = 1$ and $f_i^2 = 0$. Then for $\Lambda = 2af_1 + 2f_2$, we have $I(\Lambda) > \delta$ as required in Lemma 4.3. Note that because $\Lambda \equiv 0 \pmod 2$, for w characteristic, $\Lambda - w$ is also characteristic. Observe that because w is characteristic and $\Lambda \in B(Y)^\perp$, we have

$$\tilde{\varepsilon}(w, \Lambda, K) \equiv \varepsilon(w, K) \pmod 2.$$

For $\Lambda \in B'(Y)^\perp$, $b_{i,j,k} = 0$ unless $c(Y) + i \equiv 0 \pmod 2$ by (4.5) and thus the factor of $(1 + (-1)^{c(X)+i})$ in (4.20) equals two. Then because $K \cdot \Lambda$ and hence $b_{i,j,k} = b_{i,j,k}(\Lambda, K)$ are independent of $K \in B'(Y)$, we can write the expression for the Donaldson invariant from Lemma 4.3 as

$$D^w_Y(h^{\delta-2m} x^m) = \sum_{i+j+2k=\delta-2m} \tfrac{1}{2} b_{i,j,k} SW^w_{Y,i}(h) \langle \Lambda, h\rangle^j Q_Y(h)^k. \tag{4.21}$$

Theorem 4.11 allows us to ignore the coefficients $b_{i,j,k}$ in (4.21) with $i \leq n = c(Y) - 3$. By Proposition 4.8, we then can rewrite (4.21) as

$$D^w_Y(h^{\delta-2m} x^m) = \sum_{i+2k=\delta-2m} \sum_{K \in B'(Y)} (-1)^{\varepsilon(w,K)} \frac{(\delta - 2m)!}{2^{n+k-m} k! i!} n(K) SW'_Y(K) \langle K, h\rangle^i Q_Y(h)^k.$$

Comparing this expression with Lemma 4.2 then completes the proof of the proposition. □

References

[1] S. K. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), 257–315.

[2] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Univ. Press, Oxford, 1990.

[3] P. M. N. Feehan, P. B. Kronheimer, T. G. Leness, and T. S. Mrowka, *PU(2) monopoles and a conjecture of Mariño, Moore, and Peradze*, Math. Res. Lett. **6** (1999), 169–182, math.DG/9812125.

[4] P. M. N. Feehan and T. G. Leness, *Donaldson invariants and wall-crossing formulas. I: Continuity of gluing and obstruction maps*, submitted to a print journal, math.DG/9812060 (v3).

[5] ———, *PU(2) monopoles. III: Existence of gluing and obstruction maps*, submitted to a print journal; math.DG/9907107.

[6] ———, *PU(2) monopoles. IV: Surjectivity of gluing maps*, in preparation.

[7] ———, *An SO(3)-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, available at www.fiu.edu/~lenesst/.

[8] ———, *PU(2) monopoles and relations between four-manifold invariants*, Topology Appl. **88** (1998), 111–145, dg-ga/9709022.

[9] ———, *PU(2) monopoles. I: Regularity, Uhlenbeck compactness, and transversality*, J. Differential Geom. **49** (1998), 265–410, dg-ga/9710032.

[10] Paul M. N. Feehan and Thomas G. Leness, PU(2) *monopoles and links of top-level Seiberg-Witten moduli spaces*, J. Reine Angew. Math. **538** (2001), 57–133, math.DG/0007190.

[11] ———, PU(2) *monopoles. II. Top-level Seiberg-Witten moduli spaces and Witten's conjecture in low degrees*, J. Reine Angew. Math. **538** (2001), 135–212, dg-ga/9712005.

[12] ———, SO(3)*-monopoles: the overlap problem*, Geometry and topology of manifolds, Fields Inst. Commun., vol. 47, Amer. Math. Soc., Providence, RI, 2005, pp. 97–118.

[13] R. Fintushel and R. Stern, *Donaldson invariants of 4-manifolds with simple type*, J. Differential Geom. **42** (1995), 577–633.

[14] ———, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. **19** (1995), 145–157.

[15] ———, *Rational blowdowns of smooth* 4*-manifolds*, J. Differential Geom. **46** (1997), 181–235, alg-geom/9505018.

[16] Ronald Fintushel, Jongil Park, and Ronald J. Stern, *Rational surfaces and symplectic 4-manifolds with one basic class*, Algebr. Geom. Topol. **2** (2002), 391–402 (electronic), math.GT/0202105.

[17] R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer, Berlin, 1994.

[18] K. A. Frøyshov, *A generalized blow-up formula for Seiberg-Witten invariants*, math.DG/0604242.

[19] ———, *Monopoles over four-manifolds containing long necks, II*, math.DG/0503197v5.

[20] Kim A. Frøyshov, *Equivariant aspects of Yang-Mills Floer theory*, Topology **41** (2002), no. 3, 525–552, math.DG/9903083.

[21] P. B. Kronheimer and T. S. Mrowka, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **43** (1995), 573–734.

[22] P. B. Kronheimer and T. S. Mrowka, *Witten's conjecture and property P*, Geom. Topol. **8** (2004), 295–310 (electronic).

[23] G. Moore and E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys. **1** (1997), 298–387, hep-th/9709193.

[24] J. W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Princeton Univ. Press, Princeton, NJ, 1996.

[25] J. W. Morgan and T. S. Mrowka, *The gluing construction for anti-self-dual connections over manifolds with long tubes*, prelimary manuscript.

[26] T. S. Mrowka, *Local Mayer-Vietoris principle for Yang-Mills moduli spaces*, Ph.D. thesis, Harvard University, Cambridge, MA, 1988.

[27] V. Muñoz, *Basic classes for four-manifolds not of simple type*, Comm. Anal. Geom. **8** (2000), 653–670, math.DG/9811089.

[28] L. I. Nicolaescu, *Notes on Seiberg-Witten theory*, American Mathematical Society, Providence, RI, 2000.

[29] D. Salamon, *Spin geometry and Seiberg-Witten invariants*, Birkhäuser, Boston, to appear.

[30] C. H. Taubes, *Self-dual connections on 4-manifolds with indefinite intersection matrix*, J. Differential Geom. **19** (1984), 517–560.

[31] W. Wieczorek, *Immersed spheres and finite type for Donaldson invariants*, math.DG/9811116.

[32] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), 769–796, hep-th/9411102.

Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019
E-mail address: `feehan@math.rutgers.edu`
URL: `http://www.math.rutgers.edu/~feehan`

Department of Mathematics, Florida International University, Miami, FL 33199
E-mail address: `lenesst@fiu.edu`
URL: `http://www.fiu.edu/~lenesst`